Continuous time algorithm for Lyapunov exponents. II

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The continuous time form of the "standard" Lyapunov exponent algorithm is extended to include an imaginary part. It is shown that both the familiar constant-coefficient case and the periodic-coefficient case can be solved by this technique. However, while the algorithm is strongly stable in converging on the real parts of the system characteristic exponents, it will produce random imaginary parts. The choice of the original orthonormal matrix is critical: the special initial conditions which lead to the correct imaginary parts for the constant-coefficient and periodic-coefficient cases are found. Without a way to find the special initial orthonormal initial conditions for the general time-dependent case, the imaginary part of a Lyapunov exponent remains difficult to calculate.

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I. INTRODUCTION

The "standard" algorithm for calculating Lyapunov exponents is that of Benettin and co-workers [1,2] and Shimada and Nagashima [3]. In the preceding paper Ref. [4], hereafter referred to as paper I, we showed that this algorithm possesses a true continuous time formulation, and that the method was fully equivalent to a factorization of the system fundamental matrix into a real orthonormal matrix E and a real upper triangular matrix $\mathcal{T}$, in the form

$$\Phi(t) = E(t)\mathcal{T}(t)E^T(t_0). \tag{1}$$

It has been proven that the "standard" algorithm converges for almost any initial conditions on $E(t_0)$ to the (defined as real-valued) Lyapunov exponents of a dynamical system.

In this paper, we explore the extension of the "standard" algorithm to complex matrices E and $\mathcal{T}$. In this extension, we will be strongly guided by the solutions to the two known special cases, the constant-coefficient and periodic-coefficient linear systems. Of course, a general technique for calculating stability exponents must be capable of reproducing these well-known results as special cases of the general time-dependent linear problem.

II. TRIANGULAR DECOMPOSITION

The linearization of motion near a solution to the general dynamical system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) \tag{2}$$

leads to the *variational equations*

$$\dot{\Phi}(t, t_0) = A(t)\Phi(t, t_0) \tag{3}$$

with initial conditions $\Phi(t_0, t_0) = I$. The matrix $A(t) = \partial \mathbf{f}/\partial \mathbf{x}|_{x(t)}$ will share some of the characteristics of the reference trajectory $\mathbf{x}(t)$. It will be a constant matrix if $\mathbf{x}(t)$ is an equilibrium point, a periodic matrix if the trajectory is a periodic orbit, and a general time-dependent matrix if $\mathbf{x}(t)$ is neither of the former cases. Since the solutions to the first two of these are well known, they serve as consistency checks on any algorithm to solve the general case.

As in the preceding paper, consider factoring the fundamental matrix Φ in the form

$$\Phi(t, t_0) = E(t)\mathcal{T}(t)E^{-1}(t_0), \tag{4}$$

where now $E(t)$ and $\mathcal{T}(t)$ may be complex. Since $E^{-1}(t_0)$ is constant, the matrix $\Psi = \Phi E(t_0)$ also obeys the variational equations (3). Direct calculation of a derivative of Ψ and substitution into the variational equations then gives

$$\dot{\mathcal{T}} = \left\{E^{-1}AE - E^{-1}\dot{E}\right\}\mathcal{T} \equiv D(t)\mathcal{T}, \tag{5}$$

where the new matrix $D(t)$ is yet to be specified. We then have the pair of equations

$$\dot{\mathcal{T}} = D(t)\mathcal{T}, \tag{6}$$

$$\dot{E} = AE - ED. \tag{7}$$

In this form we have seemingly doubled the system order, but the matrix $D(t)$ is still free.

It is extremely important to ensure that one of the matrices E, $\mathcal{T}$ has strict boundedness properties. Now, there is no more bounded matrix than a unitary matrix. For such a matrix, the Hermitian transpose (denoted H) is the inverse, so if E is unitary

$$E^H E = I. \tag{8}$$

A derivative of this relation gives

$$E^H\dot{E} = -\dot{E}^H E = -\left(E^H\dot{E}\right)^H \tag{9}$$

showing that $E^H\dot{E}$ is skew Hermitian if E is unitary. A skew Hermitian matrix has zeros on its diagonal, and negatives of complex conjugates across the diagonal.

Return to Eq. (7) for $\dot{E}$, and assuming E is unitary,

write it as

$$E^H \dot{E} = E^H A E - D. \tag{10}$$

The right side above can be made skew if $D(t)$ is an upper triangular matrix. Choosing

$$d_{ii}(t) = \left(E^H A E\right)_{ii}, \tag{11}$$

$$d_{ij}(t) = \left(E^H A E\right)^*_{ji} + \left(E^H A E\right)_{ij}, \qquad j > i \tag{12}$$

will accomplish this, where $*$ is the complex conjugate, and then (7) when integrated will produce a unitary $E(t)$.

If we restrict E and $\mathcal{T}$ to be real matrices, we have the method of paper I. This is completely equivalent to the standard algorithm of Benettin and co-workers and of Shimada and Nagashima, so all of their convergence proofs (for the real-valued form) pertain to at least the real parts of the present method.

With D an upper triangular matrix, $\mathcal{T}$ also becomes upper triangular. In fact, as we will see shortly, the equation $\dot{\mathcal{T}} = D\mathcal{T}$ is a *solvable* linear system, assuming that the elements $d_{ij}(t)$ are known functions. The process is very reminiscent of the "back-substitution" portion of Gaussian elimination.

The solution to the variational equations now takes the form

$$\delta\mathbf{x}(t) = \Phi(t, t_0)\delta\mathbf{x}(t_0) = E(t)\mathcal{T}(t)E^H(t_0)\delta\mathbf{x}(t_0). \tag{13}$$

Define $\eta(t) = E^H(t)\delta\mathbf{x}(t)$. Since E is a unitary matrix, the stability properties of η must be the same as those of $\delta\mathbf{x}$. Equation (13) for the solution becomes

$$\eta(t) = \mathcal{T}(t)\eta(t_0). \tag{14}$$

Obviously, the matrix $E(t)$ mediates a coordinate transformation to a new system in which the solution to the variational equations is an upper triangular matrix. In fact, the variational equations themselves are now just $\dot{\mathcal{T}} = D(t)\mathcal{T}$, so they are upper triangular, as well.

Initial conditions can be chosen seemingly at random. Since

$$\Phi(t_0, t_0) = E(t_0)\mathcal{T}(t_0)E^H(t_0) = I, \tag{15}$$

and since E is a unitary matrix, we must have $\mathcal{T}(t_0) = I$. On the other hand, $E(t_0)$ can be any unitary matrix. This is a familiar feature of the "standard" algorithm, where convergence is assured for *any* orthonormal real-valued $E(t_0)$ matrix. But if we choose $E(t_0)$ real, it will stay real for all time, and since $A(t)$ is real valued, $D(t)$ will be real valued. It will thus be impossible to calculate a Lyapunov exponent with an imaginary part. The choice of complex $E(t_0)$ will lead to complex D and $\mathcal{T}$ matrices, and holds out the possibility of defining an imaginary part for a Lyapunov exponent.

III. REDUCTION TO KNOWN CASES

There are two known special cases of Eq. (3) which have been solved completely. These are the cases where $A(t)$ is constant, and where $A(t)$ is periodic in time. In this section we explore the reduction to upper triangular form for both of these cases.

In the constant coefficient case, it is well known that the fundamental matrix can be written as

$$\Phi(t, t_0) = e^{A(t-t_0)}. \tag{16}$$

Replace A with its triangular decomposition $A = \mathcal{E}U\mathcal{E}^H$, where $\mathcal{E}$ is unitary and U is upper triangular. Recall that when factored in this form, the eigenvalues of A appear as the diagonal elements of U. Now, expand the matrix exponential (16) via the standard Taylor's series definition as

$$\begin{aligned}\Phi(t, t_0) = {} & I + \mathcal{E}U\mathcal{E}^H(t - t_0) \\ & + \frac{1}{2!}\mathcal{E}U\mathcal{E}^H\mathcal{E}U\mathcal{E}^H(t - t_0)^2 + \cdots \\ & = \mathcal{E}\left\{I + U(t - t_0) + \frac{1}{2!}UU(t - t_0)^2 \cdots\right\}\mathcal{E}^H\end{aligned} \tag{17}$$

since $\mathcal{E}$ is a unitary matrix. But this is just

$$\Phi(t, t_0) = \mathcal{E}e^{U(t-t_0)}\mathcal{E}^H. \tag{18}$$

This is of the form (4), with $E(t) = \mathcal{E}$, constant, and $\mathcal{T} = \exp\{U(t - t_0)\}$.

Furthermore, since U is upper triangular with the eigenvalues of A along its diagonal, direct calculation from the matrix exponential gives the diagonal elements of $\mathcal{T}$ as

$$\mathcal{T}_{ii} = e^{\lambda_i(t-t_0)}. \tag{19}$$

So, the system characteristic exponents, both real and imaginary parts, are found as

$$\omega_i = \lim_{t\to\infty} \frac{1}{t - t_0} \ln \mathcal{T}_{ii} = \lambda_i. \tag{20}$$

Actually, the limit is not necessary, but is included for reasons of symmetry with later cases. This decomposition solves the constant-coefficient case "instantaneously," since no integration of (6) and (7) is needed.

The case of time periodic systems was solved by Floquet [5]. In a time periodic system, the fundamental matrix decomposes as

$$\Phi(t, t_0) = F(t)e^{J(t-t_0)}F^{-1}(t_0), \tag{21}$$

where $F(t)$ is a nonsingular periodic matrix, and J is a constant Jordan form whose diagonal elements are the Poincaré exponents.

Now, $F(t)$ can be decomposed as the product of a unitary matrix $\mathcal{E}(t)$ and an upper triangular matrix $\mathcal{U}(t)$. We proceed to show this by direct construction. Writing the columns of $F(t)$ as $\mathbf{f}_i$ and the columns of $\mathcal{E}$ as $\mathbf{e}_i$, form the matrix product

$$F(t) = \mathcal{E}\mathcal{U}. \tag{22}$$

The first column gives

$$\mathbf{f}_1(t) = u_{11}(t)\mathbf{e}_1(t), \tag{23}$$

where the scalar function $u_{11}(t)$ is the only nonzero entry in the first column of U. Since $\mathbf{e}_1$ is a normalized vector,

we immediately have $u_{11} = \| \mathbf{f}_1 \|$, and

$$\mathbf{e}_1 = \mathbf{f}_1 / \| \mathbf{f}_1 \| . \tag{24}$$

We cannot have $\| \mathbf{f}_1 \| = 0$, since F is nonsingular.

The second column of $F(t)$ is given by

$$\mathbf{f}_2(t) = u_{12}(t)\mathbf{e}_1(t) + u_{22}(t)\mathbf{e}_2(t). \tag{25}$$

Since $\mathbf{e}_1$ and $\mathbf{e}_2$ are orthogonal, we immediately find

$$u_{12} = \mathbf{e}_1 \cdot \mathbf{f}_2 \tag{26}$$

(remember the dot product of complex vectors is given by $\mathbf{e}_1 \cdot \mathbf{f}_2 = \mathbf{e}_1^H \mathbf{f}_2$). The condition that $\mathbf{e}_2$ has unit norm yields

$$u_{22} = \| \mathbf{f}_2 - (\mathbf{e}_1 \cdot \mathbf{f}_2)\, \mathbf{e}_1 \| . \tag{27}$$

Again, this can never be zero since $F(t)$ is nonsingular. Finally, we have

$$\mathbf{e}_2(t) = \{\mathbf{f}_2 - (\mathbf{e}_1 \cdot \mathbf{f}_2)\, \mathbf{e}_1\} / \| \mathbf{f}_2 - (\mathbf{e}_1 \cdot \mathbf{f}_2)\, \mathbf{e}_1 \| . \tag{28}$$

It should be obvious by now that this is the Gram-Schmidt process for orthonormalizing a set of vectors. Proceeding recursively, we do indeed find that

$$u_{ij}(t) = \mathbf{e}_i \cdot \mathbf{f}_j, \quad j > i, \tag{29}$$

$$u_{ii}(t) = \left\| \mathbf{f}_i - \sum_{j=1}^{i-1} (\mathbf{e}_j \cdot \mathbf{f}_i)\, \mathbf{e}_j \right\|, \tag{30}$$

which can never be zero, and

$$\mathbf{e}_i(t) = \left\{ \mathbf{f}_i - \sum_{j=1}^{i-1} (\mathbf{e}_j \cdot \mathbf{f}_i)\, \mathbf{e}_j \right\} u_{ii}^{-1}(t). \tag{31}$$

The nonsingularity of F guarantees that the process can be completed, while the fact that $F(t)$ is periodic in time and nonsingular shows by construction that both $\mathcal{E}(t)$ and $\mathcal{U}(t)$ are time periodic nonsingular matrices.

The Floquet solution now takes the form

$$\Phi(t, t_0) = \mathcal{E}(t)\mathcal{U}(t)e^{J(t-t_0)}\mathcal{U}^{-1}(t_0)\mathcal{E}^{-1}(t_0). \tag{32}$$

The inverse of an upper triangular matrix is also upper triangular, so this is of the form (4), with $\mathcal{E}(t)$ as the unitary matrix of Gram-Schmidt basis vectors determined from $F(t)$, and

$$\mathcal{T}(t) = \mathcal{U}(t)e^{J(t-t_0)}\mathcal{U}^{-1}(t_0) \tag{33}$$

as the upper triangular matrix. Write the elements of $\mathcal{U}(t)$ as $u_{ij}(t)$, and the elements of $\mathcal{U}^{-1}(t_0)$ as n_{ij}. Direct calculation then gives the diagonal elements of $\mathcal{T}$ as

$$\mathcal{T}_{ii} = u_{ii}(t)e^{j_i(t-t_0)}n_{ii}(t_0), \tag{34}$$

where the j_i are the Poincaré exponents. We notice that, as before,

$$\omega_i = \lim_{t\to\infty} \frac{1}{t - t_0} \ln \mathcal{T}_{ii}$$

$$= \lim_{t\to\infty} \frac{1}{t - t_0} \{\ln u_{ii}(t) + j_i(t - t_0) + \ln n_{ii}(t_0)\} = j_i \tag{35}$$

are the system characteristic exponents, with both a real and imaginary part. While (35) shows a limit, since $\mathcal{U}$ is periodic the actual values of the system characteristic exponents are available to us at the end of each period τ, since $u_{ii}(\tau) = u_{ii}(t_0) = 1/n_{ii}(t_0)$.

We have now proven that the "standard" algorithm for Lyapunov exponents, as generalized in this paper, is capable of solving both the constant-coefficient and periodic-coefficient cases. This includes any imaginary parts to the eigenvalues in the constant-coefficient case, or imaginary parts to the Poincaré exponents in the periodic case. However, the orthonormal matrix E which achieves this is *special* in both of these cases. It is found from the triangular decomposition of A in the constant-coefficient case, or the Gram-Schmidt factorization of the $F(t_0)$ matrix in the Floquet case. There is no reason to believe that an arbitrary initial $E(t_0)$ matrix will correctly calculate the imaginary parts, although the real parts are covered by the standard Lyapunov exponent convergence proofs.

IV. THE TRIANGULAR SOLUTION

We mentioned earlier that the solution to (6) for $\mathcal{T}$ can be obtained in closed form, assuming that the elements of $D(t)$ are known. Begin with the differential equation for $t_{N,N}$,

$$\dot{t}_{N,N} = d_{N,N}(t)t_{N,N}. \tag{36}$$

This integrates immediately to give

$$t_{N,N}(t) = \exp \int_{t_0}^{t} d_{N,N}(\tau)d\tau \tag{37}$$

with initial condition $t_{N,N} = 1$, since $\mathcal{T}(t_0) = I$. Now, examining the remainder of (6), we immediately notice that the above solution form holds for *all* the diagonal elements of $\mathcal{T}$:

$$t_{i,i}(t) = \exp \int_{t_0}^{t} d_{i,i}(\tau)d\tau. \tag{38}$$

The remaining equation on the next to the last row is

$$\dot{t}_{N-1,N} = d_{N-1,N-1}t_{N-1,N} + d_{N-1,N}t_{N,N}. \tag{39}$$

This is a linear time-dependent system with a known forcing term. The Green's function is

$$K_{N-1}(s,t) = \begin{cases} 0, & t < s \\ \exp\left(\int_s^t d_{N-1,N-1}(\tau)d\tau\right), & t \geq s. \end{cases} \tag{40}$$

Solving (39) by standard techniques gives

$$t_{N-1,N}(t) = \int_{t_0}^{t} K_{N-1}(s,t) d_{N-1,N}(s) K_N(t_0,s) ds, \quad (41)$$

where the initial condition of $t_{N-1,N}(t_0) = 0$ has been applied.

Generalize the Green's function K_{N-1} to $K_j(s,t)$ when the diagonal element $d_{j,j}$ is employed in (40). In the second to last row, the diagonal element has solution $t_{N-2,N-2}(t) = K_{N-2}(t_0,t)$, while the other two elements obey

$$\dot{t}_{N-2,N-1} = d_{N-2,N-2} t_{N-2,N-1} + d_{N-2,N-1} t_{N-1,N-1} \quad (42)$$

and

$$\dot{t}_{N-2,N} = d_{N-2,N-2} t_{N-2,N} + d_{N-2,N-1} t_{N-1,N} + d_{N-2,N} t_{N,N}. \quad (43)$$

These are again single variable linear systems with known forcing functions, and they both have the same Green's function, $K_{N-2}(s,t)$. Their solutions are

$$t_{N-2,N-1}(t) = \int_{t_0}^{t} K_{N-2}(s,t) \times d_{N-2,N-1}(s) K_{N-1}(t_0,s) ds \quad (44)$$

and

$$t_{N-2,N} = \int_{t_0}^{t} K_{N-2}(s,t) d_{N-2,N}(s) K_N(t_0,s) ds + \int_{t_0}^{t} K_{N-2}(s,t) d_{N-2,N-1}(s) \times \int_{t_0}^{s} K_{N-1}(\sigma,t) d_{N-1,N}(\sigma) \times K_N(t_0,\sigma) d\sigma ds. \quad (45)$$

The entire solution to (6) can be constructed in terms of convolution integrals, proceeding upwards through the $\dot{T}$ equation in a manner reminiscent of the back-substitution portion of Gaussian elimination.

V. LYAPUNOV EXPONENTS

While the upper triangular variational equations are still a coupled system, we have the solution for $\mathcal{T}$. The solution for the off-diagonal elements employs only combinations of the exponential rates encountered in solving for the diagonal elements. More to the point, since $\mathcal{T}$ is an upper triangular matrix, we know its eigenvalues. These are just the diagonal elements of $\mathcal{T}$, and those are available through (38). Thus, the average exponential rates of change of the solution η (and therefore $\delta\mathbf{x}$) are given by

$$\omega_i(t) = \frac{1}{t-t_0} \ln \mathcal{T}_{ii} = \frac{1}{t-t_0} \int_{t_0}^{t} d_{i,i}(s) ds. \quad (46)$$

In the limit as $t \to \infty$, these will then become the Lyapunov exponents generalized to include an imaginary part

$$\omega_i(\infty) = \lim_{t\to\infty} \frac{1}{t-t_0} \ln \mathcal{T}_{ii} = \lim_{t\to\infty} \frac{1}{t-t_0} \int_{t_0}^{t} d_{i,i}(s) ds. \quad (47)$$

This definition agrees with the constant-coefficient case, the periodic-coefficient case, and the real case of the "standard" Lyapunov exponent algorithm as given in paper I.

Returning to the pair of systems (6) and (7), we see that it is not necessary to solve for $\mathcal{T}(t)$ at all, if we are only interested in calculating the Lyapunov exponents. It is only necessary to propagate $E(t)$, and perform the N integrals in (46). These are easily done themselves as differential equations. Since the underlying reference trajectory must be propagated as well, we must solve $3N+2N^2$ simultaneous real-valued differential equations.

VI. NUMERICAL EXPERIMENTS

We have conducted extensive testing of this method on several dynamical systems, including constant-coefficient and periodic-coefficient systems, as well as some generalized time-dependent systems with chaotic structure. As was reported in paper I, the equations of motion for the unitary matrix E, Eqs. (6) and (12), are not necessarily numerically stable. That is, while (6) and (12) do mathematically produce a unitary E matrix when given a unitary $E(t_0)$, not all solutions to these differential equations are unitary. Numerical errors will inevitably cause $E(t)$ to depart slightly from a perfect unitary condition. Then (6) and (12) may produce an exponential departure from the desired unitary solution. Or the numerical errors may decay exponentially, also. Equations (6) and (12) are nonlinear in E, and have their *own* stability properties.

To avoid this, the program we have constructed stores only a portion of E including the diagonal elements and elements above the diagonal. At each time step, the last vector $\mathbf{e}_N$ is renormalized to eliminate any numerical error. The missing last component of the next to last vector $\mathbf{e}_{N-1}$ is reconstructed from the requirements that it be (i) orthogonal to $\mathbf{e}_N$ and (ii) a unit vector. That is,

$$\mathbf{e}_N \cdot \mathbf{e}_{N-1} = 0, \quad \mathbf{e}_{N-1} \cdot \mathbf{e}_{N-1} = 1 \quad (48)$$

uniquely determine the missing last element of $\mathbf{e}_{N-1}$. The last two missing elements of $\mathbf{e}_{N-2}$ are determined from

$$\mathbf{e}_N \cdot \mathbf{e}_{N-2} = 0, \ \mathbf{e}_{N-1} \cdot \mathbf{e}_{N-2} = 0, \ \mathbf{e}_{N-2} \cdot \mathbf{e}_{N-2} = 1, \quad (49)$$

and so forth. We are thus propagating the $(N^2+N)/2$ upper triangular portion of Eq. (6), with the savings in the number of differential equations being somewhat offset by the need to reconstruct the rest of E at each integration step. The resulting E matrix will be rigorously unitary, however.

(Notice that since $\mathcal{T}$ is upper triangular and E is unitary, the apparent doubling of the number of differential equations is not really correct. As we have implemented the above method, we have only N "extra" differential equations. If the final element of each vector $\mathbf{e}_i$ were determined from the normalization requirement, this would eliminate all superfluous differential equations, at the expense of introducing a sign ambiguity for the last element. We have chosen not to introduce this ambiguity.)

For constant-coefficient systems, the factorization of Sec. III does indeed produce the correct system characteristic exponents if the correct initial unitary matrix $\mathcal{E} = E(t_0)$ is used. However, when random complex unitary $E(t_0)$ are employed, the result is different. For example, the constant-coefficient system

$$A = \begin{Bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -13 & -17 & -5 \end{Bmatrix} \tag{50}$$

has eigenvalues $\omega_i = -1, \ -2 \pm 3i$. These are returned by the method of the present paper if the special initial conditions for $E(t_0)$ are used. Figure 1 shows the results of our algorithm with 20 different random $E(t_0)$ matrices as initial conditions. The purely real root is calculated correctly in all cases, as are the real parts of the complex-conjugate roots. However, the imaginary parts lie at random values between $-3i \leq \text{Im } \omega_i \leq 3i$. This pattern has been observed in every other system examined. The real parts are correctly calculated for *any* initial $E(t_0)$, while the imaginary parts are only correct if the "correct" $E(t_0)$ is used.

For the time-periodic case we have also explored several systems. For the limit cycle in the Van der Pol equation the Poincaré exponents are purely real, and are correctly calculated by the present method even for random initial conditions $E(t_0)$. Of course, the "correct" $E(t_0)$ calculated from Floquet theory enables them to be produced with a much shorter integration, since they can be extracted at the end of one period. We have also constructed periodic orbits in the restricted problem of three bodies, a well known Hamiltonian system. For one example we have extensively studied, the true Poincaré exponents are $\omega_i = \pm 0, \ \pm 0.664153i$. As shown in Fig. 2, the

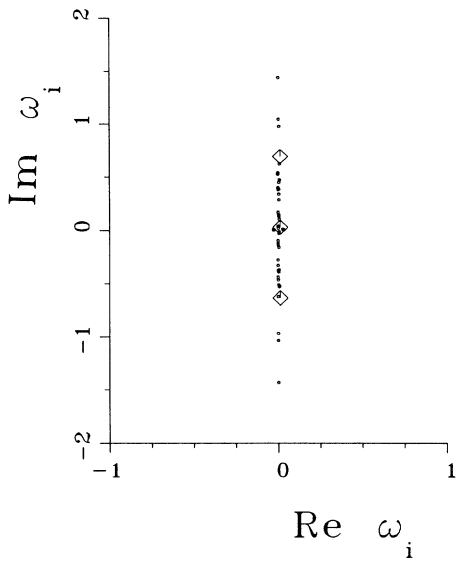

FIG. 2. Calculated ω_i for 20 cases with random initial $E(t_0)$ for a periodic orbit in the restricted problem of three bodies. True values (diamonds on the plot) are $\omega_i = \pm 0, \ \pm 0.66415i$.

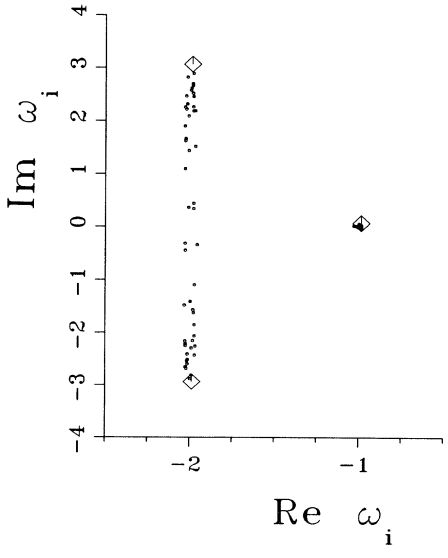

FIG. 1. Calculated ω_i for 20 cases of random initial $E(t_0)$ for a constant-coefficient system with eigenvalues $\omega_i = -1, \ -2 \pm 3i$. The true characteristic exponents are indicated by diamonds.

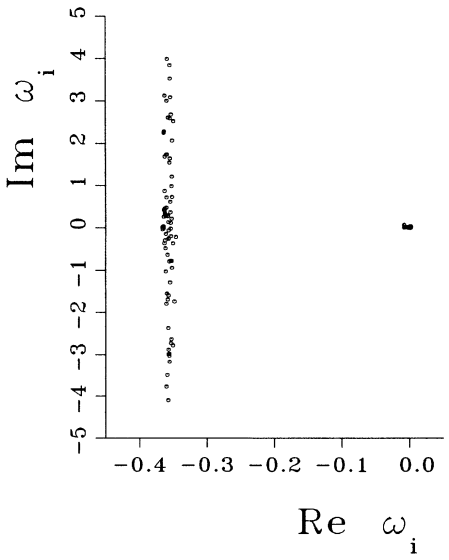

FIG. 3. Calculated ω_i for 20 cases with random initial $E(t_0)$ for the coupled Van der Pol system. One pair of roots shows nonzero imaginary parts.

zero roots are correctly calculated by the present technique, as are the zero real parts of the purely imaginary Poincaré exponents. The imaginary parts, though, scatter randomly (although in a bounded region) for random initial $E(t_0)$. Notice that the true values of the Poincaré exponents do not lie at the extremes of the calculated values from the numerical experiments.

Finally, for chaotic systems we have examined both the Lorenz attractor and the coupled Van der Pol equations studied by Kapitaniak and Steeb [6]. This system has equations of motion

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= a(1-x_1^2)x_2 - x_1^3 + b(\sin x_5 + x_3), \\ \dot{x}_3 &= x_4, \\ \dot{x}_4 &= a(1-x_3^2)x_4 - x_3^3 + b(\sin x_5 + x_1), \\ \dot{x}_5 &= \omega. \end{aligned} \tag{51}$$

We have used $a = 0.2$ and $\omega = 4$, with initial conditions $x_1 = 1$, $x_2 = 0$, $x_3 = 0$, $x_4 = 0$, and $x_5 = 0$. But we have picked $b = 5$, which is below the transition to chaos described in [6]. Figure 3 again shows 20 different calculations of Lyapunov exponents using different random initial $E(t_0)$. Real parts were confirmed both by the standard algorithm and by the method of paper I. One pair of roots with real parts near $\mathrm{Re}\omega_i \approx -0.36$ has conjugate imaginary parts, while the other pair is purely real. No "true" values are shown, since no comparison method is available.

VII. DISCUSSION AND CONCLUSIONS

We have shown that the continuous time version of the standard method, as presented in paper I, can be extended to include an imaginary part. Furthermore, this extension agrees with both the constant-coefficient and periodic-coefficient cases *when the correct initial conditions are used for the unitary matrix* $E(t_0)$. Numerical experiments confirm the success of this extension, as well as emphasizing the necessity of correct initial conditions for E. Otherwise, the algorithm calculates the real parts correctly, but the imaginary parts are random. The numerical work also confirms that some general time-dependent systems show nonzero imaginary parts.

[1] G. Benettin, L. Galgani, A. Giorgilli, and J. Strelcyn, C. R. Acad. Sci. Paris **286A**, 431 (1978).
[2] G. Benettin, L. Galgani, A. Giorgilli, and J. Strelcyn, Meccanica **15**, 9 (1980).
[3] I. Shimada and T. Nagashima, Prog. Theor. Phys. **61**, 1605 (1979).
[4] W. Wiesel, preceding paper, Phys. Rev. E **47**, 3686 (1993).
[5] M. G. Floquet, Ann. Ecole Normale Sup. Ser. 2 **12**, 47 (1883).
[6] T. Kapitaniak and W.-H. Steeb, Phys. Lett. A **152**, 33 (1991).